LESSON

14 Graphs

Here is where everything comes together. We know how to find the domain, how to identify asymptotes, and how to plot points. With the help of the sign diagrams from the previous lesson, we shall be able to tell where a function is increasing and decreasing, and where it is concave up and down.

Quite simply, where the derivative is positive, the function is increasing. The derivative gives the slope of the tangent line at a point, and when this is positive, the function is heading upward, viewed from left to right. When the derivative is negative, the function slopes downward and decreases.

When the second derivative is positive, the function is concave up. This is because the second derivative says how the first derivative is changing. If the second derivative is positive, then the slopes are increasing. If the slopes, from left to right, increase from 22, to 21, to 0, to 1, to 2, and so on, then the graph must curve like the one in Figure 14.1. In other words, the curve must be concave up.

Similarly, if the second derivative is negative, the function curves downward like the one in Figure 14.2 and is concave down.

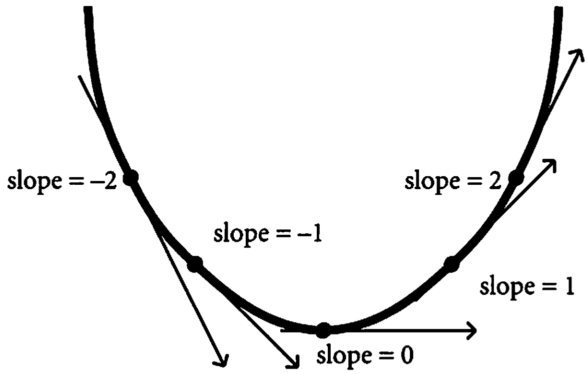

Figure 14.1

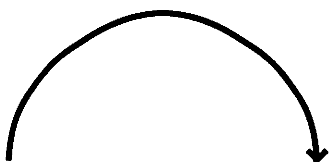

Figure 14.2

The concavity governs the shape of the graph, depending on whether the function $f(x)$ is increasing or decreasing. If $f(x)$ is increasing and concave up (thus, both $f'(x)$ and $f''(x)$ are positive), then the graph has the shape shown in Figure 14.3.

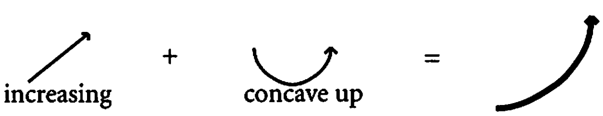

Figure 14.3

If $f(x)$ is increasing and concave down (thus, $f'(x)$ is positive and $f''(x)$ is negative), then the graph has the shape shown in Figure 14.4.

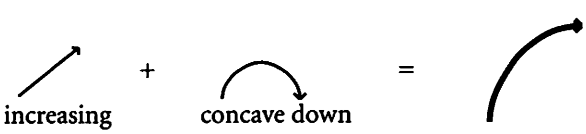

Figure 14.4

If $f(x)$ is decreasing and concave down (thus, both $f'(x)$ and $f''(x)$ are negative), then the graph has the shape of the one in Figure 14.5.

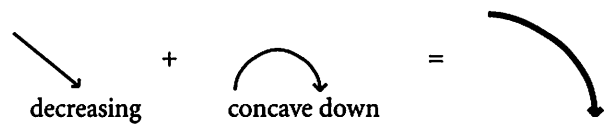

Figure 14.5

If $f(x)$ is decreasing and concave up (thus, $f'(x) < 0$ and $f''(x) > 0$), the graph has the shape of the one in Figure 14.6.

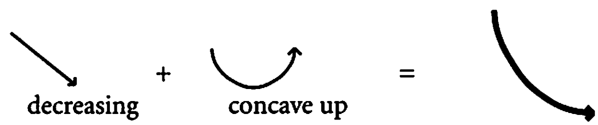

Figure 14.6

Example

Graph $f(x) = x^3 + 6x^2 - 15x + 10$.

Solution

This function is defined everywhere and thus has no vertical asymptotes. Because $\lim_{x\to\infty} x^3 + 6x^2 - 15x + 10 = \infty$ and $\lim_{x\to-\infty} x^3 + 6x^2 - 15x + 10 = -\infty$, there are no horizontal asymptotes.

The derivative $f'(x) = 3x^2 + 12x - 15 = 3(x^2 + 4x - 5) = 3(x + 5)(x - 1)$ is zero at $x = -5$ and $x = 1$. To form the sign diagram, we test: $f'(-6) = 21$, $f'(0) = 15$, and $f'(2) = 21$. **Note:** These points were chosen arbitrarily. Any point less than -5 will give the same information as the value $x = -6$, for instance, and any point between -5 and -1 will give the same information as the value at $x = 0$. Thus, the sign diagram for $f'(x)$ is shown in Figure 14.7.

Increasing or Decreasing

Remember, the sign of $f'(x)$ determines whether $f(x)$ is increasing or decreasing.

Note: We use $f'(x)$ to see if the graph is increasing or decreasing, but $f(x)$ to find the y-value at a point.

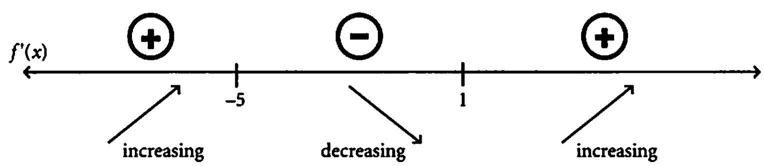

Figure 14.7

Because the function increases up to $x = -5$ and then decreases immediately afterward, there is a local maximum at $x = -5$. The corresponding y-value is $y = f(-5) = 110$. Thus, $(-5,110)$ is a local maximum. Similarly, the graph goes down to $x = 1$ and then goes up afterward, so $x = 1$ is a local minimum. The corresponding y-value is $f(1) = 2$, so $(1,2)$ is a local minimum. $f'(x)$ and $f(x)$

A guideline for identifying local minimum and maximum points is shown in Figure 14.8.

The second derivative is $f''(x) = 6x + 12 = 6(x + 2)$, which is zero at $x = -2$. If we test the sign at $x = -3$ and $x = 0$, we get $f''(-3) = -6$ and $f''(0) = 12$. Thus, the sign diagram for $f''(x)$ is as shown in Figure 14.9.

Figure 14.8

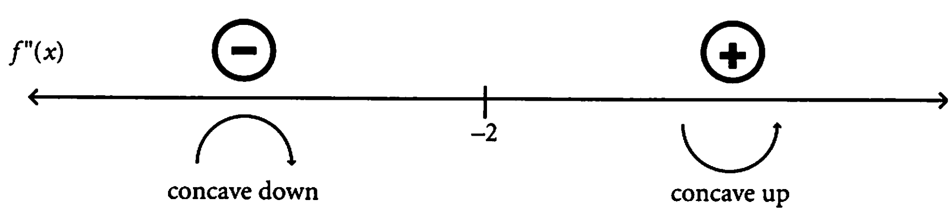

Figure 14.9

Clearly $x = -2$ is a point of inflection, because this is where the concavity switches from concave down to concave up. The y-value of this point is $f(-2) = 56$.

Before we draw the axes for the Cartesian plane, we should consider the three interesting points we have found: the local maximum at $(-5,110)$, the local minimum at $(1,2)$, and the point of inflection at $(-2,56)$. If our x-axis runs from $x = -10$ to $x = 10$, and our y-axis runs from 0 to 120, then all of these can be plotted easily (see Figure 14.10).

Example

Graph $g(x) = \dfrac{x+3}{x-2}$.

Solution

The domain is $x \neq 2$. There is a vertical asymptote at $x = 2$. The sign diagram or $g(x)$ is shown in Figure 14.11.

Thus, $\lim\limits_{x\to 2^-} \dfrac{x+3}{x-2} = -\infty$ and $\lim\limits_{x\to 2^+} \dfrac{x+3}{x-2} = \infty$.

Because $\lim\limits_{x\to\infty} \dfrac{x+3}{x-2} = 1$ and $\lim\limits_{x\to-\infty} \dfrac{x+3}{x-2} = 1$, there

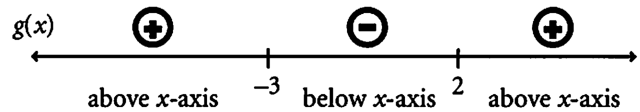

Figure 14.11

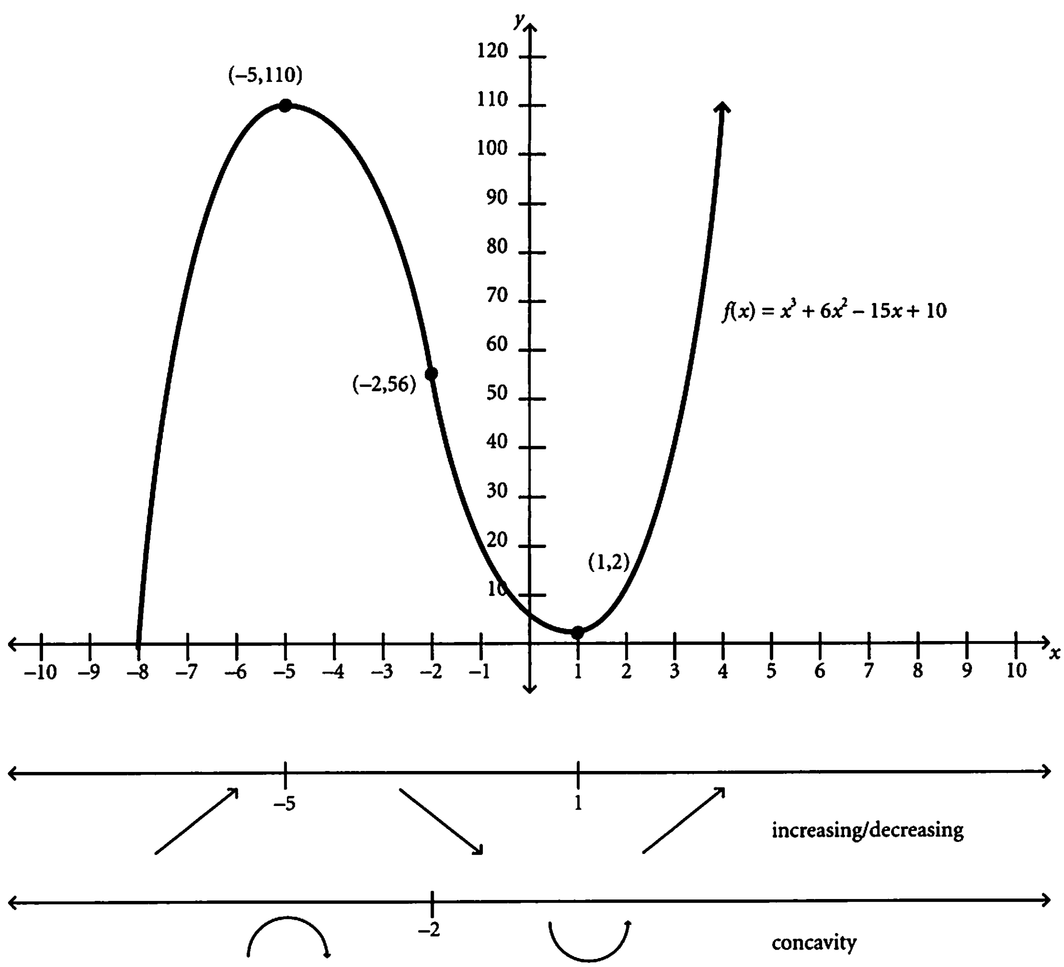

Figure 14.10

is a horizontal asymptote at $y = 1$, both to the left and to the right. The derivative $g'(x) = \dfrac{1 \cdot (x-2) - 1 \cdot (x+3)}{(x-2)^2} = \dfrac{-5}{(x-2)^2}$ has the sign diagram shown in Figure 14.12.

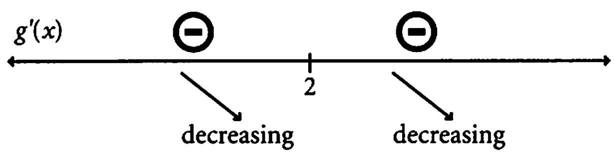

Figure 14.12

The second derivative $g''(x) = \dfrac{10}{(x-2)^3}$ has sign diagram shown in Figure 14.13.

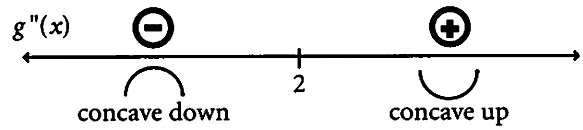

Figure 14.13

Because we have no points plotted at all, it makes sense to pick one or two to the left and right of the vertical asymptote at $x = 2$. At $x = 1$, $g(1) = -4$, so $(1,-4)$ is a point. At $x = 3$, $g(3) = 6$, so (3,6) is another point. At $x = -3$, $g(-3) = 0$, so $(-3,0)$ is another nice point to know. Judging by these, it will be useful to have both the x- and y-axes run from -10 to 10.

To graph $g(x)$, it helps to start with the points and the asymptotes as shown in Figure 14.14.

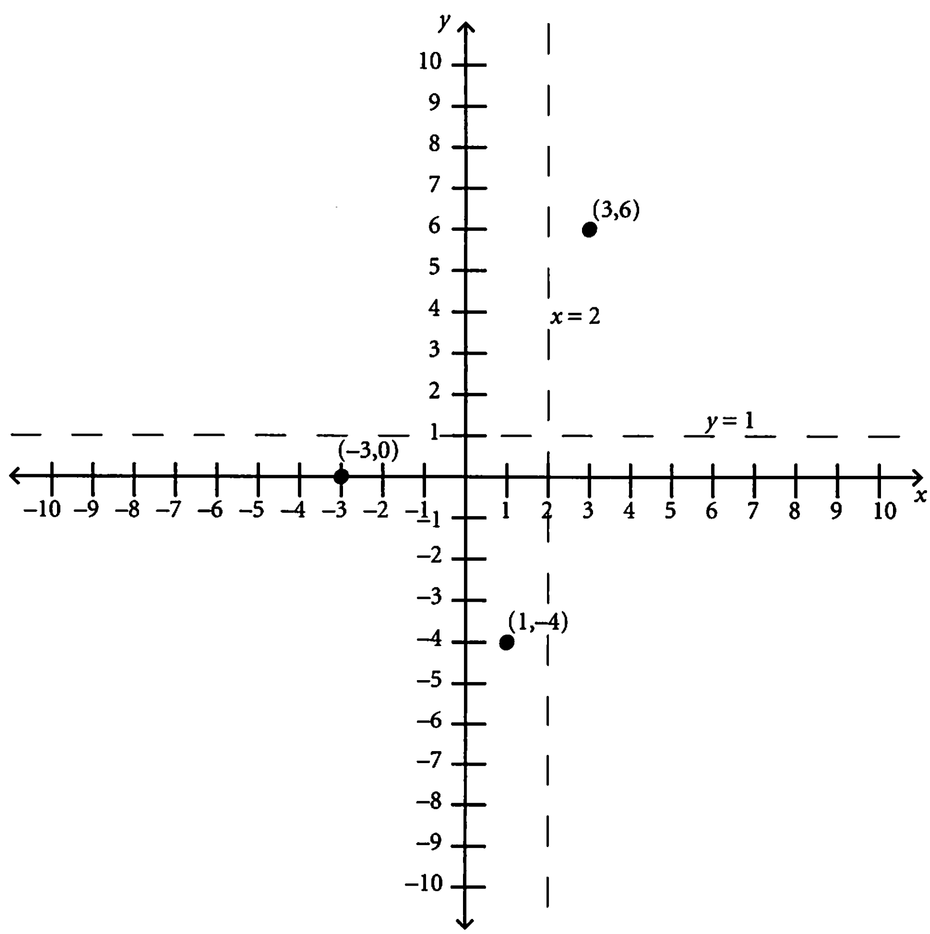

Figure 14.14

Then we establish the shapes of the lines through these points using the concavity and the intervals of decrease (see Figure 14.15).

Example

Graph $h(x) = \dfrac{x^2 + 1}{x^2 - 1}$.

Solution

To start, $h(x) = \dfrac{x^2 + 1}{x^2 - 1} = \dfrac{x^2 + 1}{(x + 1)(x - 1)}$. Thus, $h(x)$ is undefined with a vertical asymptote at $x = 1$ and $x = -1$. The sign diagram for $h(x)$ is shown in Figure 14.16.

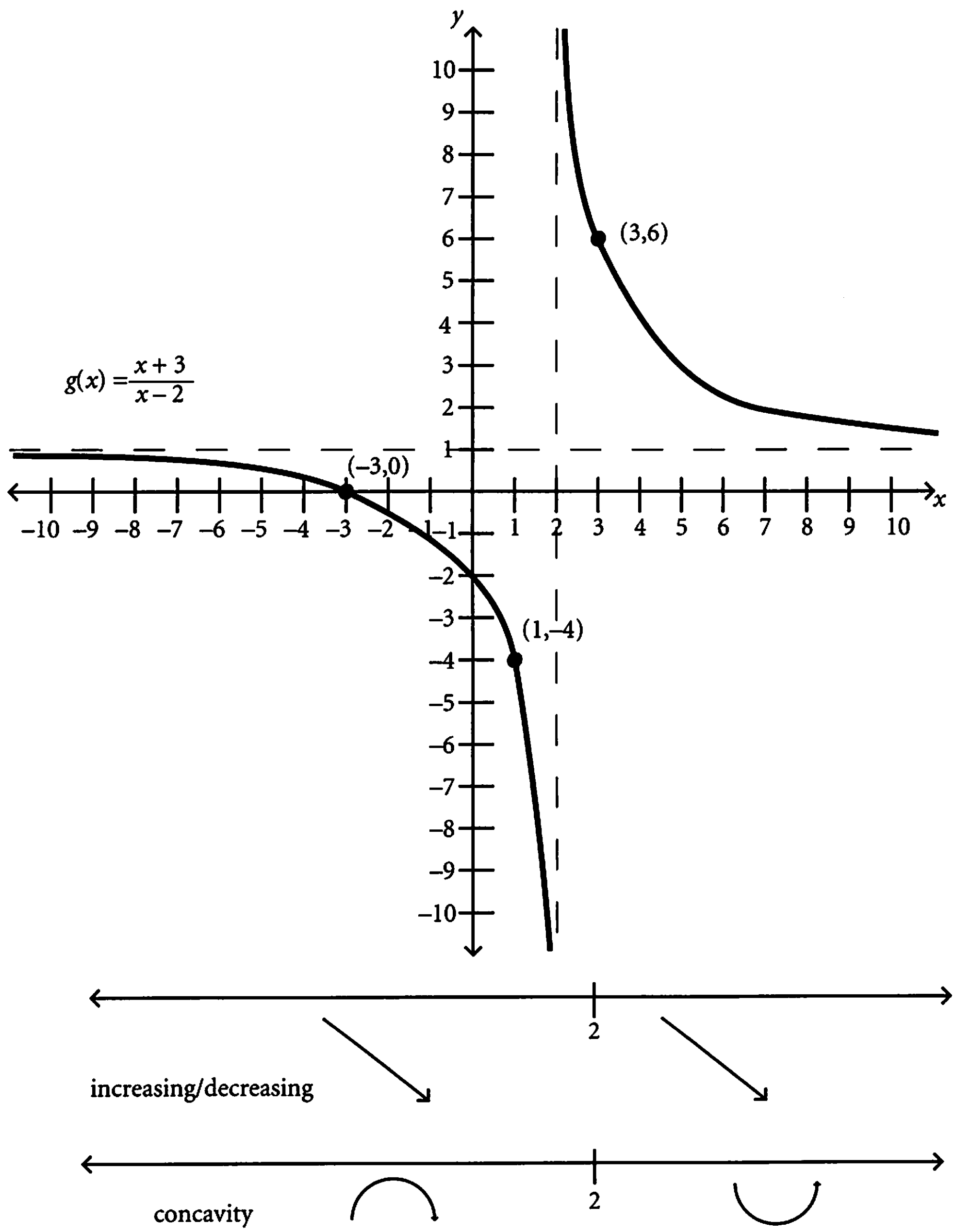

Figure 14.15

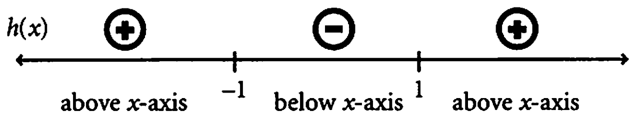

Figure 14.16

Note: $x^2 + 1$ can never be zero. The limits at the vertical asymptotes are thus:

$$\lim_{x\to -1^-} \frac{x^2+1}{x^2-1} = \infty$$

$$\lim_{x\to -1^+} \frac{x^2+1}{x^2-1} = -\infty$$

$$\lim_{x\to 1^-} \frac{x^2+1}{x^2-1} = -\infty$$

$$\lim_{x\to 1^+} \frac{x^2+1}{x^2-1} = \infty$$

Because $\lim_{x\to\infty} \frac{x^2+1}{x^2-1} = 1$ and $\lim_{x\to -\infty} \frac{x^2+1}{x^2-1} = 1$, there is a horizontal asymptote at $y = 1$.

The derivative is as follows:

$$h'(x) = \frac{2x(x^2-1) - 2x(x^2 1)}{(x^2-1)^2} = \frac{-4x}{(x-1)^2(x+1)^2}$$

It has the sign diagram shown in Figure 14.17. This indicates that there is a local maximum at $x = 0$. The corresponding y-value is $y = h(0) = -1$.

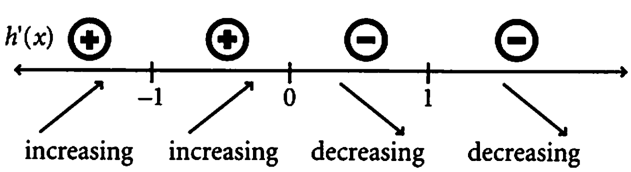

Figure 14.17

The second derivative is as follows:

$$h''(x) = \frac{-4(x^2-1)^2 - 2(x^2-1)\cdot 2x(-4x)}{(x^2-1)^4}$$

$$= \frac{-4(x^2-1) - 2\cdot 2x(-4x)}{(x^2-1)^3}$$

$$= \frac{12x^2+4}{(x^2-1)^3} = \frac{12x^2+4}{(x-1)^3(x+1)^3}.$$

The sign diagram is shown in Figure 14.18. It looks like there ought to be points of inflection at $x = -1$ and $x = 1$, but these are asymptotes not in the domain, so there are no points where the concavity changes.

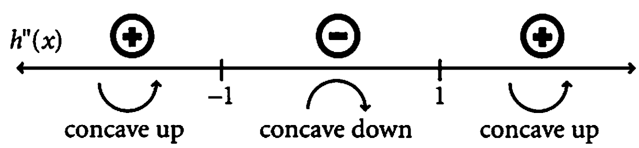

Figure 14.18

Before we graph the function, it will be useful to have a few more points. When $x = -2$, then $y = h(-2) = \frac{5}{3}$ and when $x = 2$, $y = h(2) = \frac{5}{3}$ as well. Thus, it will be useful to have the x- and y-axes run from -3 to 3. We start with just the points and asymptotes (see Figure 14.19).

Then we add in the actual curves, guided by the concavity and the intervals of increase and decrease (see Figure 14.20).

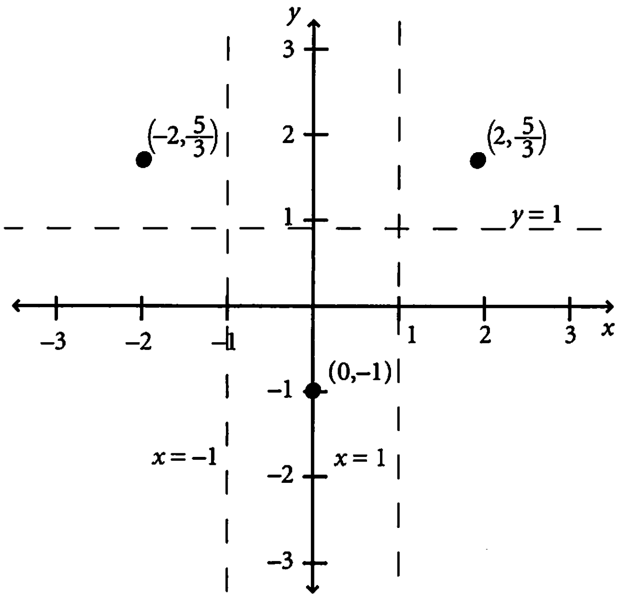

Figure 14.19

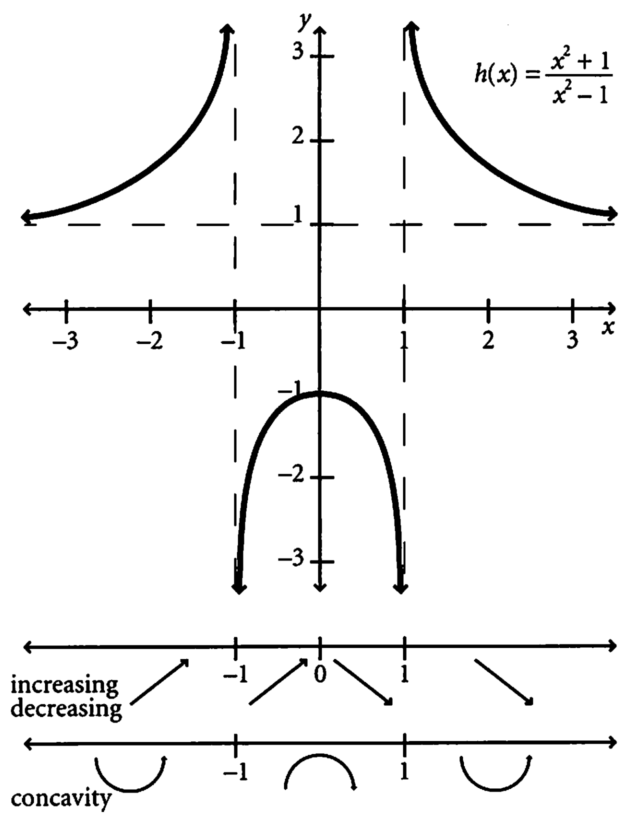

Figure 14.20

▶ Practice

Use the asymptotes, concavity, and intervals of increase and decrease to graph the following functions.

1. $f(x) = x^2 - 30x + 10$
2. $g(x) = -4x - x^2$
3. $h(x) = 2x^3 - 3x^2 - 36x + 5$
4. $k(x) = 3x - x^3$
5. $f(x) = x^4 - 8x^3 + 5$
6. $g(x) = \frac{x}{x+2}$
7. $h(x) = \frac{1}{x^2-9}$
8. $k(x) = \frac{x}{x^2-1}$
9. $j(x) = \frac{x^2+1}{x}$
10. $f(x) = \frac{x}{x^2+1}$